Definition: (sign)

The **sign** of $\sigma \in S_n$,

denoted by $\text{sign}(\sigma)$,

is defined as

$$\boxed{\text{sign}(\sigma) = \begin{cases} 1, & \sigma \text{ decomposes as an even number of transpositions} \\ -1, & \sigma \text{ decomposes as an odd number of transpositions} \end{cases}}$$

Properties:

1) If e = the identity bijection,

$$\text{sign}(e) = 1$$

2) If $\sigma, \tau \in S_n$,

$$\text{sign}(\sigma \circ \tau) = \text{sign}(\sigma) \cdot \text{sign}(\tau)$$

3) $\text{sign}(\sigma^{-1}) = \text{sign}(\sigma)$

4) If σ is a transposition,

$\text{sign}(\sigma) = -1$.

Definition: (determinant)

Let $A \in M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$)

If $A = (a_{i,j})_{i,j=1}^{n}$,

define the determinant

$\det(A)$ to be

$$\boxed{\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sign}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \right)}$$

Compactly,

$$\det(A) = \sum_{\sigma \in S_n} \left(\text{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

" $\prod$ " = product

Example 1: (2×2)

Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$.

Two permutations on $\{1, 2\}$

: $\sigma(1) = 1$, $\sigma(2) = 2$

$\tau(1) = 2$, $\tau(2) = 1$

σ = identity, $\text{sign}(\sigma) = 1$

τ = transposition, $\text{sign}(\tau) = -1$

Then by definition,

det(A)

$$= a_{1,\sigma(1)} a_{2,\sigma(2)} \underbrace{-}_{\text{sign}(\tau)} a_{1,\tau(1)} a_{2,\tau(2)}$$

$$= \boxed{a_{1,1} a_{2,2} - a_{1,2} a_{2,1}} \checkmark$$

Theorem: (determinant props)

Let $A, B \in M_n(\mathbb{C})$, $\alpha \in \mathbb{C}$

1) $\det(I_n) = 1$

2) $$\boxed{\begin{aligned}\det(AB) &= \det(A)\det(B) \\ &= \det(B)\det(A)\end{aligned}}$$

3) $\det(\alpha A) = \alpha^n \det(A)$

proof:

1) $\det(I_n)$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(\prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

but in the identity matrix,

$$a_{i,j} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

So if $\exists\ j,\ 1 \le j \le n,$

with $\sigma(j) \neq j,$

then

$$\prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$= a_{j,\sigma(j)} \prod_{\substack{1 \le i \le n \\ i \neq j}} a_{i,\sigma(i)}$$

$= 0$ since $a_{j,\sigma(j)} = 0$.

We get

$$\det(I_n)$$

$$= \operatorname{sign}(e) \prod_{i=1}^{n} a_{i,e(i)}$$

$$= 1 \cdot \prod_{i=1}^{n} a_{i,i}$$

$$= 1$$

3) Let $A=(a_{i,j})_{i,j=1}^{n}$.

Then $\alpha A=(\underbrace{\alpha a_{i,j}}_{c_{i,j}})_{i,j=1}^{n}$

and $\det(\alpha A)$

$$=\sum_{\sigma\in S_n}\operatorname{sign}(\sigma)\prod_{i=1}^{n}c_{i,\sigma(i)}$$

$$=\sum_{\sigma\in S_n}\operatorname{sign}(\sigma)\prod_{i=1}^{n}\alpha a_{i,\sigma(i)}$$

$$= \alpha^n \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$= \alpha^n \det(A)$$

2) Let $A = (a_{i,j})_{i,j=1}^{n}$, $B = (b_{i,j})_{i,j=1}^{n}$

$$\det(A) \cdot \det(B)$$

$$= \left(\sum_{\tau \in S_n} \operatorname{sign}(\tau) \prod_{i=1}^{n} a_{i,\tau(i)} \right) \cdot \left(\sum_{\gamma \in S_n} \operatorname{sign}(\gamma) \prod_{j=1}^{n} b_{j,\gamma(j)} \right)$$

$$= \sum_{\tau \in S_n} \sum_{\gamma \in S_n} \operatorname{sign}(\tau) \operatorname{sign}(\gamma) \prod_{i=1}^{n} a_{i,\tau(i)} \, b_{i,\gamma(i)}$$

Now,

$$(AB)_{i,\sigma(i)} = \sum_{k=1}^{n} a_{i,k} b_{k,\sigma(i)},$$

so

$$\det(AB) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} \left(\sum_{k=1}^{n} a_{i,k} b_{k,\sigma(i)} \right)$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \sum_{1 \le k_1, k_2, \ldots, k_n \le n} \prod_{i=1}^{n} a_{i,k_i} b_{k_i,\sigma(i)}$$

Now consider, for a fixed σ and fixed n-tuple $\{k_1, k_2, \ldots, k_n\}$,

$$\operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,k_i} b_{k_i,\sigma(i)}.$$

If $\exists$ $1 \leq s, t \leq n$, $s \neq t$, with $\boxed{k_s = k_t}$,

then rewrite

$$\operatorname{Sign}(\sigma)\, a_{s,k_s} b_{k_s,\sigma(s)} a_{t,k_t} b_{k_t,\sigma(t)} \prod_{\substack{1\le i\le n \\ i\ne s,t}} a_{i,k_i} b_{k_i,\sigma(i)}$$

Then with δ the transposition

$$\boxed{\delta(s)=t,\ \delta(t)=s},$$

and $\varphi=\sigma\circ\delta$,

$$\varphi(i)=\sigma(i)\ \forall\ 1\le i\le n,\ i\ne s,t$$

and
$$\varphi(s)=(\sigma\circ\delta)(s)=\sigma(t)$$
$$\varphi(t)=(\sigma\circ\delta)(t)=\sigma(s).$$

With the same n-tuple $\{k_1, k_2, \ldots, k_n\}$, we examine the term in the sum

$$\operatorname{sign}(\varphi) \prod_{i=1}^{n} a_{i,k_i} b_{k_i,\varphi(i)}$$

$$= \operatorname{sign}(\sigma \circ \delta)\, a_{s,k_s} b_{k_s,\varphi(s)} a_{t,k_t} b_{k_t,\varphi(t)} \prod_{\substack{i=1 \\ i \neq s,t}}^{n} a_{i,k_i} b_{k_i,\varphi(i)}$$

$$= \operatorname{sign}(\sigma) \underbrace{\operatorname{sign}(\delta)}_{-1} a_{s,k_s} b_{k_t,\sigma(s)} a_{t,k_t} b_{k_s,\sigma(t)} \prod_{i=1}^{n} a_{i,k_i} b_{k_i,\sigma(i)}$$

But since $k_s = k_t$, this is

$$= -\operatorname{Sign}(\sigma)\, a_{s,k_s} b_{k_s,\sigma(s)} a_{t,k_t} b_{k_t,\sigma(t)} \prod_{i=1}^{n} a_{i,k_i} b_{k_i,\sigma(i)}$$

Therefore the term in the sum associated to $\delta = \sigma \circ \varphi$ cancels with the term associated with σ. So a term in the sum survives if and only if

$$\{k_1, k_2, \dots, k_n\} = \{1, 2, \dots, n\}.$$

But then the association $i \mapsto k_i$ defines an element $\tau \in S_n$, so we may rewrite the sum $\det(AB)$ as

$$\sum_{\sigma \in S_n} \sum_{\tau \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i,\tau(i)} b_{\tau(i),\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i,\tau(i)} b_{i,\sigma\circ\tau^{-1}(i)}$$

See next page

To see why this last equality is true, write

$$\prod_{i=1}^{n} a_{i,\tau(i)} b_{\tau(i),\sigma(i)}$$

$$= \prod_{i=1}^{n} a_{i,\tau(i)} \prod_{j=1}^{n} b_{\tau(j),\sigma(j)}$$

Since τ is a bijection, for each $j \in \mathbb{N}$, $\exists$ a unique $k \in \mathbb{N}$ with

$$\tau(k) = j.$$

Then $\quad k = \tau^{-1}(j)$

and

$$\prod_{j=1}^{n} b_{\tau(j),\sigma(j)} = \prod_{k=1}^{n} b_{k,\sigma\circ\tau^{-1}(k)}$$

Finally, we again combine:

$$\prod_{i=1}^{n} a_{i,\tau(i)} \prod_{k=1}^{n} b_{k,\sigma\circ\tau^{-1}(k)}$$

$$= \prod_{i=1}^{n} a_{i,\tau(i)} b_{i,\sigma\circ\tau^{-1}(i)}$$

Now

$$\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma \circ \tau^{-1} \circ \tau)$$
$$= \operatorname{sign}(\sigma \circ \tau^{-1}) \operatorname{sign}(\tau)$$

Setting $\gamma = \sigma \circ \tau^{-1}$,

$\det(AB)$

$$= \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\tau(i)} b_{i,\sigma \circ \tau^{-1}(i)}$$

$$= \sum_{\gamma \in S_n} \sum_{\tau \in S_n} \operatorname{sign}(\gamma) \operatorname{sign}(\tau) \prod_{i=1}^{n} a_{i,\tau(i)} b_{i,\gamma(i)}$$

$$= \det(A) \det(B)$$